18-819F: Introduction to Quantum Computing
47-779/47-785: Quantum Integer Programming & Quantum Machine Learning

Review of Linear Algebra II

Lecture 02

2021.09.07.

Agenda

- Dirac notation for complex linear algebra
  - State vectors
  - Linear operators – matrices
  - Hermitian operators
  - Unitary operators
  - Projection operators and measurement
- Tensors
  - Basis vectors for tensor product space
  - Composite quantum systems
  - Conception of tensor products as quantum entanglement

Dirac Notation for Linear Algebra

- We previously defined the inner (dot) product of two vectors u and v as

$$u.v = (u, v),$$

where it was understood that u is a *row* vector, which is better written as the transpose, u^T.

- This bracket notation was specialized by P.A.M Dirac, using angle brackets, to

$$u.v = u^T.v =< u||v > = < u|v >;$$

- An ordinary row vector is now defined as: $< u| \ = [u_1 \quad u_2 \quad \cdots \quad u_n]$, where $< \ |$ is the "bra" and an ordinary column vector is $|v > = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, where $| \ >$ is the "ket" so that the inner (dot) product is formed by the "bra-ket" $< u|v >$;
- The "bra" implicitly tells us to take the complex conjugate and the transpose of u.

Dirac Notation for Linear Algebra

- Having defined the notation, one can now use it to write most of ordinary vector and matrix algebra;

- The outer product of two vectors, which we previously wrote as

$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [v_1 \quad v_2 \quad \dots \quad v_n] = \begin{bmatrix} u_1 v_1 & \dots & u_1 v_n \\ \vdots & \dots & \vdots \\ u_n v_1 & \dots & u_n v_n \end{bmatrix}$ Eqn. (3.1), is a *matrix* that can be written succinctly as $uv^T = |u >< v|$ (we prove this later) Eqn. (3.2);

- The outer product is often called a *dyad* and is a *linear operator*, i.e., a matrix.

Dyad in 3D Vector Space

- In 3D linear algebra, we said we can write a vector $\vec{v}$ as a linear combination of the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$, where v_1, v_2, v_3 are the scaling coefficients

$$\left.\begin{aligned} \vec{v} &= v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \\ \vec{v} &= \hat{e}_1(\hat{e}_1^T.\vec{v}) + \hat{e}_2(e_2^T.\vec{v}) + \hat{e}_3(e_3^T.\vec{v}) \\ \vec{v} &= (\hat{e}_1\hat{e}_1^T + \hat{e}_2\hat{e}_2^T + \hat{e}_3\hat{e}_3^T)\vec{v} \end{aligned}\right\}$$

- From the 2nd equation above we have: $v_1 = \hat{e}_1^T.\vec{v}$, $v_2 = \hat{e}_2^T.\vec{v}$, $v_3 = e_3^T.\vec{v}$;

- From the 3rd equation above we have $\hat{1} = \hat{e}_1\hat{e}_1^T + \hat{e}_2\hat{e}_2^T + \hat{e}_3\hat{e}_3^T = \sum_{i=1}^{3} \hat{e}_i\hat{e}_i^T$;

- So $\sum_{i=1}^{3} e_i e_i^T = \hat{1}$ is the closure (completeness) relationship for the 3D vector space.

Relationship of Dyad Operator to $\hat{1}$

- In Hilbert space, a state vector $|v>$ can be written as a linear combination of the basis vectors $|1>, |2>, \ldots |n>$ with complex coefficients $c_1, c_2, \ldots c_n$

$$\left.\begin{array}{lcl} |v> & = & c_1|1> + c_2|2> + \cdots + c_n|n> \\ |v> & = & |1>(<1|v>) + |2>(<2|v>) + \cdots + |n>(<n|v> \\ |v> & = & (|1><1| + |2><2| + \cdots |n><n|)|v> \end{array}\right\}$$

- As in ordinary vector space, the coefficients c_i are "how much" of each basis vector is in the state vector $|v>$, which is given by the inner product (or the projection of the state vector onto each basis vector);
- The left-hand side of the last equation is equal to the right-hand side if

$$\hat{1} = |1><1| + |2><2| + \cdots + |n><n| = \sum_{i=1}^{n} |i><i|$$

Operators as Matrices

- We know that the closure relationship for a basis set of vectors $|i>$ for a vector space is
$$\sum_i |i><i| = \hat{1}$$
- Any operator $\hat{A}$ in this vector space can therefore be written as
$$\hat{A} = \hat{1}\hat{A}\hat{1} = \sum_i \sum_j |i><i|\hat{A}|j><j| = \sum_{ij} <i|\hat{A}|j> |i><j| = \sum_{ij} A_{ij}|i><j|$$
- Note that $< i|\hat{A}|j = A_{ij}$;
- Suppose the basis set is the standard basis: $|1> = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |2> = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} |3> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \ldots |n> = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$
- It follows then that $\hat{A} = \sum_{ij} A_{ij} |i><j| = \begin{bmatrix} A_{11} & A_{12} & \ldots \\ A_{21} & A_{22} & \\ \vdots & & \ddots \end{bmatrix}$ is a matrix (QED).

Example of a Special Operator: the Hamiltonian

- Every quantum system has a Hamiltonian operator, $\hat{H}$, representing the total system energy.

- Suppose the state space of the system is spanned by the basis set $|u_j>$, then

$$\hat{H}|u_j> = \lambda_j|u_j>, \text{ with } \lambda_j \text{ being the eigenvalues;}$$

- We can therefore write

- $$\begin{aligned} \hat{H} &= \hat{1}\hat{H}\hat{1} = \left(\sum_j |u_j><u_j|\right)\hat{H}\left(\sum_k |u_k><u_k|\right) \\ &= \sum_{jk}|u_j><u_j|u_k>\lambda_k<u_k| \\ &= \sum_{jk}|u_j>\delta_{jk}\lambda_k<u_k| \end{aligned}$$

- The last equation tells us that $\hat{H} = \sum_k \lambda_k |u_k><u_k|$, which is a matrix as we saw previously.

- Hence $\hat{H} = \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{bmatrix}$; apparently an operator is diagonal in its own eigen basis.

Inner Product and the Length of a Vector

- Given the state vector $|\psi>=\sum_n c_n|\varphi_n>$, where φ_n are basis vectors, one can use the inner product to determine the length of the state vector state$|\psi>$, thus

$$<\psi|\psi>=\sum_m\sum_n c_m^* c_n<\varphi_m|\varphi_n>=\sum_n|c_n|^2;$$

- The length of the state vector is therefore $||\psi||=\sqrt{<\psi|\psi>}$;

- For two state vectors $|U>=\sum_m a_m|u_m>$ and $|V>=\sum_n b_n|v_n>$, the inner product is

$$<U|V>=\sum_m\sum_n a_m^* b_n<u_m|v_n>=\sum_n a_n^* b_n$$

Linear operators

- A linear operator $\hat{O}$ can transform vectors (states) in the following manner

$$\hat{O}(\alpha|\psi > +\beta|\varphi >) = \alpha\hat{O}\left|\psi > +\beta\hat{O}\right|\varphi > \text{ Eqn. (3.3);}$$

- The expression above is simply following ordinary rules of linear algebra (for addition, multiplication, and distributivity);

- Note that $|\psi >$ and $|\varphi >$ are “state” vectors and α and β are complex (scalars);

- Remember that the operator $\hat{O}$ is simply a matrix that transforms one vector to another as we discussed before.

Average (Expectation) Value

- Given an operator A and the state φ that is a linear combination of a set of basis vectors, u_i, we can write

$$|\varphi> = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow |\varphi >= c_1|u_1> + c_2|u_2> + \cdots + c_n|u_n> \text{ Eqn. (3.4);}$$

- We can compute the average (*expectation*) result of operating on the state as

$$\langle A \rangle = \langle \varphi | A | \varphi \rangle = \sum_i \sum_j c_i^* c_j < u_i | A | u_j >;$$

- Define $< u_i|\hat{A}|u_j > \equiv a_{ij}$ as matrix elements; since we must have $i = j$

$$\langle \hat{A} \rangle = \langle \varphi | A | \varphi \rangle = \sum_i c_i^* a_{ii} c_j = \sum_i a_i |c_i|^2 = \sum_i a_i P_i \quad \text{Eqn. (3.5).}$$

Matrix Elements

- For the state vector $|\varphi>$, which we said could be written as a linear combination of basis vectors in a Hilbert space,

$$|\varphi >= c_1|u_1> + c_2|u_2> + \cdots + c_n|u_n> \quad \text{Eqn. (3.6)},$$

- We can perform the operation $\hat{A}|\varphi>$, to get another state vector; the inner product of this resulting vector with another basis vector $|u_j>$ is given by

$$< u_j|\hat{A}|\sum_k c_k|u_k> = \sum_k < u_j|\hat{A}|u_k> c_k \text{ Eqn. (3.7)};$$

- The expression $< u_j|\hat{A}|u_k> = a_{jk}$ is a *number* called the *matrix element* that expresses the coupling of state $|u_k>$ to state $|u_j>$; (see previous slide on expectation);

- When $j = k$, then $< u_j|\hat{A}|u_k> = \langle\hat{A}\rangle$ is just the expectation value of the operator $\hat{A}$.

Hermitian Operators

- A Hermitian conjugate of an operator A is obtained by complex conjugating and then taking the transpose of the operator

$$A^{\dagger} = (A^{*})^{T};$$

- Some properties of Hermitian conjugates

$$\begin{aligned} (AB)^{\dagger} &= B^{\dagger}A^{\dagger} \\ (|\psi>)^{\dagger} &= <\psi| \\ (A|\psi>)^{\dagger} &= <\psi|A^{\dagger} \\ (AB|\psi>)^{\dagger} &= <\psi|B^{\dagger}A^{\dagger} \\ (\alpha A)^{\dagger} &= \alpha^{*}A^{\dagger} \end{aligned}$$

- An operator is Hermitian if $A = A^{\dagger}$; this is the complex analog of the symmetric matrix;

- The eigenvalues of a Hermitian operator are always real.

Unitary Operators

- Recall that we defined an inverse of an operator (matrix) A through $AA^{-1} = I$, where I is the identity matrix;

- An operator A is *unitary* if its adjoint (transpose) is equal to its inverse: $A^{\dagger} = A^{-1}$;

- Unitary operators are usually denoted by the symbol U; from the definition above we get

$$UU^{\dagger} = U^{\dagger}U = I \quad \text{Eqn. (3.8);}$$

- An operator A is *normal* if $AA^{\dagger} = A^{\dagger}A$ Eqn. (3.9);

- Hermitian and unitary operators are normal.

Trace of an Operator

- Given an operator A as the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

we define the trace as the sum of the diagonal elements, $Tr(A) = a_{11} + a_{22}$;

- For a general operator A and a basis vectors u_i, we can write

$$Tr(A) = \sum_i < u_i|A|u_i > = \sum_i a_{ii} \quad \text{Eqn. (3.10);}$$

Some properties of the trace:

- Trace is cyclic, i.e., $Tr(ABC) = Tr(CAB) = Tr(BCA)$;
- Trace is independent of basis vectors, $Tr(A) = \sum_i < u_i|A|u_i > = \sum_i < v_i|A|v_i >$;
- If operator A has eigenvalues λ_i , then $Tr(A) = \sum_i \lambda_i$;
- Trace is linear, meaning that $Tr(\alpha A) = \alpha Tr(A)$; and $Tr(A + B) = Tr(A) + Tr(B)$;

Projection Operators

- We defined earlier that the outer product of two vectors, $|\psi >$ and $|\varphi >$, is an operator, A (matrix),

$$|\psi >< \varphi| = A \quad \text{Eqn. (3.11);}$$

- When the outer product is between a *normalized* vector with itself, for example, $|\chi >$ and $|\chi >$, we get a special operator called the projection operator,

$$P = |\chi >< \chi| \quad \text{Eqn. (3.12);}$$

- The projection operator in the example above projects any vector it operates on to the direction of $|\chi >$;

Properties of Projection Operators

- For any normalized vector, $|u_1>$, we have $<u_1|u_1>=1 \Longrightarrow P=|u_1><u_1|$;
- $P^\dagger=|u_1><u_1|=P$;
- P is Hermitian since $P|u_1>=|u_1><u_1|u_1>=|u_1>$;
- $P^2=P$ since we can show that:
- $P^2=P|u_1><u_1|=|u_1><u_1|u_1><u_1|=|u_1><u_1|=P$;
- The identity I is a projection operator: $I^\dagger=I, I^2=I$;
- For $|\psi>=\sum_i c_i u_i \Longrightarrow \sum_i P_i|\psi>=\sum_i|u_i><u_i|\psi>=\sum_i c_i|u_i>=|\psi>$;
- If λ_i are eigenvalues of A with eigenvectors, $|a_i>$, then $A|a_i>=\lambda_i|a_i>$; we can therefore write the operator $A=\sum_i A|a_i><a_i|=\sum_i \lambda_i|a_i><a_i|$, where we have used the spectral decomposition theorem discussed earlier;

Measurement

- Measurement is an important concept in quantum mechanics and is best understood through the projection operator, P;

- Assume a quantum mechanical system is initially in the state described by the vector $|\Psi >$;

- A measurement of the quantity $\mathcal{A}$ for the system is represented by the operator $\hat{A}$ whose eigenvectors are given by

$$\hat{A}|\psi_i > = \lambda_i|\psi_i > \qquad \text{Eqn. (3.13);}$$

- After a measurement, the system is projected onto the direction of the eigenvector $|\psi_i >$,

$$P_i|\Psi > = |\psi_i >< \psi_i|\Psi > = < \psi_i|\Psi > |\psi_i > \qquad \text{Eqn.(3.14);}$$

- The probability of the outcome of the measurement is given by $|< \psi_i|\Psi >|^2$

Another Perspective on Quantum Measurement

- Imagine a quantum system is in state $|\varphi> = \sum_i c_i|u_i>$. If a measurement of an observable $\mathcal{O}$ associated with the system is made, what is the result of the measurement?

- An observable $\mathcal{O}$ is represented by an (Hermitian) operator $\hat{\mathcal{O}}$; therefore, a measurement of the system could result in any one of the eigenvalues of the system, given by

$$\hat{\mathcal{O}}|u_i> = \lambda_i|u_i>;$$

The question of interest is: which one? We need to compute the probability. Since the sum of all probabilities of measuring any one of the eigenvalues is 1, we can write

$$1 = <\psi|\psi> = <\psi|1|\psi> = <\psi|\sum_i |u_i><u_i|\psi> = \sum_i <\psi|u_i><u_i|\psi>$$

- The last expression simply says that $\sum_i <\psi|u_i><u_i|\psi> = \sum_i |c_i|^2 = 1;$

- The probability of the measurement yielding one of the λ_i eigenvalues is therefore

$$P_i = |<u_i|\psi>|^2 = |c_i|^2.$$

Generalization of the Gram-Schmidt Process for Hilbert space

- The Gram-Schmidt orthogonalization process for $3D$ space we discussed previously can be generalized to nD space following the same steps as before;

- For a vector space U spanned by the basis vectors $\{u_1, u_2, \dots u_n\}$ with a defined inner product, we can calculate an orthogonal basis v_i by following the (algorithm) steps:

1. $|v_1 > = |u_1 >$ Eqn. (3.15);

2. $|v_2 > = \left|u_2 > - \frac{<v_1|u_2>}{<v_1|v_1>}\right| u_1 >$ Eqn. (3.16);

$\vdots$

n. $\left|v_n > = \left|u_n > - \frac{<v_1|u_n>}{<u_1|v_1>}\right| v_1 > - \frac{<v_2|u_n>}{<v_2|v_2>}\right| v_2 > - \cdots - \frac{<v_{n-1}|u_n>}{<v_{n-1}|v_{n-1}>} | v_{n-1} >$ (3.17).

Generalized Vectors and Matrices: Tensors

- We can combine two vectors u in $\mathbb{R}^m$ and v in $\mathbb{R}^n$ to create another vector in the combined $\mathbb{R}^{mn}$ space by the operation of multiplication; the new vector is called a *tensor product*;

- Supposed $u = \begin{bmatrix}1\\2\end{bmatrix}$ and $v = \begin{bmatrix}3\\4\\5\end{bmatrix} \Longrightarrow u \otimes v = \begin{bmatrix}1.3\\1.4\\1.5\\2.3\\2.4\\2.5\end{bmatrix} = \begin{bmatrix}3\\4\\5\\6\\8\\10\end{bmatrix}$;

- We have produced a new vector $u \otimes v$ in $U \otimes V$ in the space $\mathbb{R}^2 \otimes \mathbb{R}^3$;

- NB: only the tensor (Kronecker) product is of interest to us at this time.

Basis Vectors for Tensor Product Space

- Any vector in $V \otimes U$ is a weighted sum of basis vectors; note that V is in $\mathbb{R}^m$ and U is in $\mathbb{R}^n$;
- The basis for $V \otimes U$ is the set of all vectors of the form $v_i \otimes u_j$ for $i = 1$ to m and $j = 1$ to n;
- The basis set for $V \otimes U$ is illustrated on the figure on the right with 6 basis vectors;
- Observe that $v \otimes u$ is the outer product of v and u:

$$v \otimes u = vu^T;$$

- We now learn apparently that any $m \times n$ matrix can be reshaped into an $mn \times 1$ vector and vice versa!

	u_1	u_2
v_1	$v_1 \otimes u_1$	$v_1 \otimes u_2$
v_2	$v_2 \otimes u_1$	$v_2 \otimes u_2$
v_3	$v_3 \otimes u_1$	$v_3 \otimes u_2$

Tensors and Matrix Reshaping

- Suppose we have the vectors $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, we can form the tensor product

$$v \otimes u = vu^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 1.4 & 1.5 \\ 2.4 & 2.5 \\ 3.4 & 3.5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix};$$

- We can reshape the matrix into a vector or the vector into a matrix as below

$$\begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 \\ 5 \\ 8 \\ 10 \\ 12 \\ 15 \end{bmatrix}.$$

Tensors and Composite States

- Suppose we have the two basis vectors: $|0>$ in $H_1$and $|1>$ in H_2, where H_1 and H_2 are the respective Hilbert spaces. What is the basis for the combined Hilbert space

$$H = H_1 \otimes H_2?$$

- Using what we have learned so far, the basis for the combined Hilbert space can be written as all the possible products of the basis states for H_1 and H_2, thus

$$|0> \otimes |0> = |00>, |0> \otimes |1> = |01>, \; |1> \otimes |0> = |10>, |1> \otimes |1> = |11>;$$

- In a more general example, given the state $\psi = a_1|x> + a_2|y>$ with basis vectors $|x>$ and $|y>$ in H_1 and the state $|\varphi> = b_1|u> + b_2|v>$ with the basis vectors $|u>$ and $|v>$ in H_2 one can create the combined Hilbert space $H_1 \otimes H_2$ where the new state is described by

$$|\chi> = \psi \otimes \varphi = (a_1|x> + a_2|y>) \otimes (b_1|u> + b_2|v>)$$

$$\Rightarrow |\chi> = a_1 b_1 |x> \otimes |u> + a_1 b_2 |x> \otimes |v> + a_2 b_1 |y> \otimes |u> + a_2 b_2 |y> \otimes v>.$$

Tensors and Process-State Duality

- We learned that $v \otimes u$ can also be a matrix, which is a mathematical *process* (operator) representing a linear transformation, but we also found that $v \otimes u$ is an abstract vector, which represents a *state* of a system;

- Evidently matrices encode processes and vectors encode states;

- We can therefore view the tensor product $V \otimes U$ as either a process or a state simply by reshaping the numbers as a rectangle or a list;

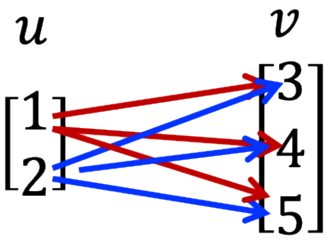
- By generalizing the idea of processes to higher dimensional arrays, we get tensors that can be constructed to create tensor networks;

Graphical Perspective of a Tensor Product

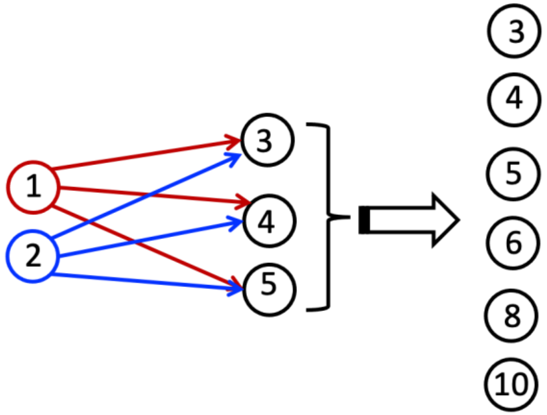
- We previously created a tensor $u \otimes v$ out of the two vectors u and v as illustrated below;

- Another way to look at this process is by examining the interactions between the “nodes” that perform the multiplication or computation of the product; this perspective leads to the graphical illustration alongside the vector tensor multiplication;

- The graphical perspective is reminiscent of *artificial neural network interconnections*; in fact, this perspective illuminates a computational scheme known as “*tensor flow*”.



$$\Rightarrow \quad u \otimes v: \begin{bmatrix} 1.3 \\ 1.4 \\ 1.5 \\ 2.3 \\ 2.4 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 8 \\ 10 \end{bmatrix}$$



Operators and Tensors

- Given and operator A that acts on $|\psi>$ in H_1 and an operator B that acts on $|\varphi>$ in H_2, we can create an operator $A \otimes B$ that acts on the vectors in the combined vector space $H_1 \otimes H_2$, thus

$$(A \otimes B)(|\psi> \otimes |\varphi>) = A|\psi> \otimes B|\varphi>;$$

- If the two operators A and B are given as the matrices

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, we calculate the tensor product as

$$A \otimes B = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$ Eqn. (3.18).

Describing Quantum Particle Interactions with Tensors

- Quantum particles, such as atoms, ions, electrons, and photons, can be manipulated to perform some computing; the catch is that we must describe these particles interact with one another;
- We may need to know what these particles are doing, what their status (state) is, in how many ways they are interacting, or what the probability of their being in certain states is; the state of a quantum particle can be described by a complex *unit* vector in $\mathbf{C}^n$;
- The combined state of two particles, one described by a *basis vector* v_1, in $\mathbf{C}^n$and the other by v_2 in $\mathbf{C}^n$ is a tensor product of their individual Hilbert spaces; thus

$$\mathbf{C}^n \otimes \mathbf{C}^n;$$

- As we discussed previously, the resulting basis vectors of the combined Hilbert space can be thought of as a matrix, called the *density matrix*, ρ;
- For N particles, density matrix ρ is on $C^n \otimes C^n \otimes C^n \ldots \otimes C^n = (C^n)^N$ Eqn. (3.19).

SVD of a Complex Matrix : Schmidt Decomposition

- When matrix M is complex, as in quantum mechanical operators, then $M = U\Lambda V^{\dagger}$;

- If we write the columns of U as u_i and those of V as v_i, then

$$M = \begin{bmatrix} . & | & . \\ . & u_i & . \\ . & | & . \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \sigma_i & \\ & & \ddots \end{bmatrix} \begin{bmatrix} -- & v_i & -- \end{bmatrix} = \sigma_1 u_1 v_1^{\dagger} + \sigma_2 u_2 v_2^{\dagger} + \cdots + \sigma_r u_r v_r^{\dagger} \text{ Eqn. (3.20);}$$

- Recall that $uv^T = u \otimes v = |u >< v|$; because of this, (3.20) above can be written as (3.21) below;

- Matrix M can be written as state a $|\psi > = \sigma_1 u_1 \otimes v_1 + \sigma_2 u_2 \otimes v_2 + \cdots + \sigma_r u_r \otimes v_r$

$$\Rightarrow |\psi > = \sum_{i=1}^{r} \sigma_i u_i \otimes v_i \text{ Eqn. (3.21).}$$

SVD of a Complex Matrix and Quantum Entanglement

- When we decompose a complex matrix M we can write it as state vector

$$|\psi> = \sum_{i=1}^{r} \sigma_i u_i \otimes v_i \qquad \text{Eqn. (3.21);}$$

- The *singular values* σ_i are renamed the *Schmidt coefficients*, and the rank of the matrix M (which is the number of non-zero singular values) r, is renamed the Schmidt rank;

- Quantum state vector $|\psi>$ now represents *entanglement* when the Schmidt rank $r > 1$; however, the system is not entangled when the rank is not larger than 1;

- Quantum entanglement is a resource in quantum computing and communication.

Summary

- Reviewed complex linear algebra relevant for quantum mechanics (used in quantum computing and communication applications);
  - Complex vectors
  - Complex matrices

- Introduced the notion of tensor product
  - A way to interconvert between vectors and matrices and vice versa
  - Briefly introduced (mathematically) the idea of entanglement